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(2,0)-supersymmetric sigma models and almost complex structures

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ABSTRACT

We find a new class of (2,0)-supersymmetric two-dimensional sigma models with torsion and target spaces almost complex manifolds extending similar results for models with (2,2) supersymmetry. These models are invariant under a new symmetry which is generated by a Noether charge of Lorentz weight one and it is associated to the Nijenhuis tensor of the almost complex structure of the sigma model target manifold. We compute the Poisson bracket algebra of charges of the above (2,0)-and (2,2)-supersymmetric sigma models and show that it closes but it is not isomorphic to the standard (2,0) and (2,2) supersymmetry algebra, respectively. Examples of such (2,0)- and (2,2)-supersymmetric sigma models with target spaces group manifolds are also given. In addition, we study the quantisation of the (2,0)-supersymmetric sigma models, compute the anomalies of their classical symmetries and examine their cancellation. Furthermore, we examine the massive extension of (2,0)-supersymmetric sigma models with target spaces almost complex manifolds, and study the topological twist of the new supersymmetry algebras.

1. Introduction

The (p,q) supersymmetry algebra in two dimensions is

$$\{S^I_+, S^J_+\} = 2\delta^{IJ} T_{\ddagger}, \qquad \{S^{I'}_-, S^{J'}_-\} = 2\delta^{I'J'} T_{=}\ , \qquad \{S^I_+, S^{I'}_-\} = Z^{II'}\ , \tag{1.1}$$

where $\{S^I_+; I = 0, \ldots, p-1\}$ are the 'left' supersymmetry charges, $\{S^{I'}_-; I' = 0, \ldots, q-1\}$ are the 'right' supersymmetry charges, $T_{\ddagger} = E + P$, $T_{=} = E - P$, E is the energy, P is the momentum and $Z^{II'}$ are central charges of the algebra⋆. The subscripts in the above charges denote the Lorentz weight of the charge; for example S^I_+ has Lorentz weight $\frac{1}{2}$, $S^{I'}_-$ has Lorentz weight $-\frac{1}{2}$ and $Z^{II'}$ has Lorentz weight zero. The central charges are zero for massless theories, *i.e* theories without a parameter with dimension that of a mass. In the following, we will discuss only *massless* models unless it is otherwise stated. The supersymmetry algebra (1.1), up to an isomorphism, is the one expected for a supersymmetric theory from the Haag-Lopuszanski-Sohnius theorem [1].

Realisations of the supersymmetry algebra (1.1) in terms of Poisson bracket algebras of charges of supersymmetric sigma models in two dimensions have been extensively studied in the literature [2, 3,4]. The main observation regarding these realisations is that (p,q) supersymmetry imposes restrictions on the geometry of the sigma model manifolds. To illustrate this, we will summarise the geometry of the target spaces of massless (2,2)- and (2,0)-supersymmetric sigma models. The geometry of the target space $\mathcal{M}$ of two-dimensional supersymmetric sigma models with (2,2) supersymmetry depends upon the properties of two (1,1) tensors I and J on $\mathcal{M}$ which appear naturally in the (2,0) and (0,2) supersymmetry transformations of the fields. For (2,2)-supersymmetric sigma models without Wess-Zumino term, or 'torsion', $I = J$, I is a complex structure which is covariantly constant with respect to the Levi-Civita connection $\Gamma(g)$ of the sigma model metric g and

⋆ This algebra can be enhanced by adding additional generators that rotate the supersymmetry charges. This possibility will not be considered here.

g is hermitian with respect to I, *i.e.* the sigma model manifold $\mathcal{M}$ is Kähler [2]. These models admit a conventional superfield formulation in terms of chiral superfields. For (2,2)-supersymmetric sigma models with Wess-Zumino term b and on-shell closure of the algebra of (2,2) supersymmetry transformations, I and J are again complex structures, but $I \neq J$, the sigma model metric g is hermitian with respect to both complex structures, and I and J are covariantly constant with respect to the connections $\Gamma^{(+)} \equiv \Gamma(g) + H$ and $\Gamma^{(-)} \equiv \Gamma(g) - H$ with torsion, respectively; the torsion $H = \frac{3}{2}db$. A superfield formulation for some of these models has been proposed in ref. [5]. Examples of sigma models with on-shell (2,2) supersymmetry are the supersymmetric extensions of the WZW models [6]. The algebra of supersymmetry transformations of (2,2) sigma models with torsion closes off-shell provided that, in addition to the conditions for on-shell closure mentioned above, the complex structures I and J commute, $IJ = JI$. In this case there is a conventional (2,2) superfield formulation of the theory [3].

Sigma models with (2,0) supersymmetry are naturally associated with a pair of a Riemannian manifold $\mathcal{M}$, the sigma model manifold, and a vector bundle $\mathcal{E}$ over $\mathcal{M}$. The geometry of $\mathcal{M}$ and $\mathcal{E}$ of (2,0)-supersymmetric sigma models with on-shell supersymmetry also depends upon the properties of a (1,1) tensor I on $\mathcal{M}$. On-shell closure of the algebra of (2,0) supersymmetry transformations requires that the tensor I is a complex structure which is covariantly constant with respect to $\Gamma^{(+)}$ connection, the sigma model metric is hermitian with respect to I and the curvature of a connection of the $\mathcal{E}$ bundle is an (1,1) tensor with respect to I [7], *i.e.* $\mathcal{M}$ is a hermitian manifold and the complexified bundle of $\mathcal{E}$ is holomorphic. For off-shell closure of the algebra of (2,0) supersymmetry transformations, in addition to I and the conditions mentioned above for on-shell closure of the (2,0) supersymmetry transformations, a new (1,1) tensor $\hat{I}$ is required on the fibre of the vector bundle $\mathcal{E}$ such that $\hat{I}$ is a complex structure, the bundle space of $\mathcal{E}$ is a complex manifold with respect to the pair of complex structures $(I, \hat{I})$ and a fibre metric of $\mathcal{E}$ is hermitian with respect to $\hat{I}$ [4]. The (2,0)-supersymmetric sigma models with off-shell supersymmetry admit a conventional superfield formulation

in terms of constrained superfields [4].

More recently, new (2,2)-supersymmetric sigma models with torsion have been considered in refs.[8,9,10]. The tensors I and J of these models are *almost* complex structures that are covariantly constant with respect to the connections $\Gamma^{(+)}$ and $\Gamma^{(-)}$, respectively, and the sigma model metric is hermitian with respect to both almost complex structures I and J. These new models differ from the standard (2,2) supersymmetric sigma models mentioned above because their action is invariant under transformations generated by the Nijenhuis tensors of the almost complex structures I and J; we will call these symmetries Nijenhuis symmetries. The algebra of supersymmetry and Nijenhuis transformations of the above models closes [10] provided that their parameters are constant. Note that if the torsion H is zero the Nijenhuis tensors of I and J are zero as well and the sigma model manifold is Kähler.

Our main interest in this paper is to find the conditions for the existence of sigma models with (2,0) supersymmetry and target spaces which are *not* complex manifolds. We will show that such models have target spaces almost complex manifolds and new symmetries associated with the Nijenhuis tensor of the almost complex structures. We will compute the Poisson bracket algebra of the charges of the (2,0)-supersymmetric sigma models with target space almost complex manifolds and we will show that the non-vanishing Poisson brackets are the following:

$$\{\overset{0}{S}_+, \overset{0}{S}_+\} = 2T_{\mp}, \qquad \{S^1_+, S^1_+\} = 2(T_{\mp} + N_{\mp}) \ , \tag{1.2}$$

where the charges $\overset{0}{S}_+$, S^1_+, $T_{\mp}$ are as in (1.1) and $N_{\mp}$ is the charge that generates the Nijenhuis symmetry of the model. We will also compute the Poisson bracket algebra of charges of (2,2)-supersymmetric sigma models with target spaces almost complex manifolds and show that it is isomorphic to two commuting copies of (1.2). The supersymmetry algebra (1.2) of the above (2,0)-supersymmetric sigma models is *not* isomorphic to the corresponding standard supersymmetry algebra (1.1). We will give examples of (2,2)- and (2,0)-supersymmetric sigma models with Nijenhuis

symmetries and target spaces group manifolds. Furthermore, we will also study the quantisation of (2,0)-supersymmetric sigma models and compute the anomalies associated with the (2,0) supersymmetry and Nijenhuis transformations using the descent equations. We will also examine the cancellation of these anomalies and we will show that some of (2,0))-supersymmetric sigma models with classical Nijenhuis symmetries and target spaces group manifolds are anomaly free. In addition, we will extend the above results to massive (2,0)-supersymmetric sigma models and we will examine a topological twist of the supersymmetry algebra (1.2).

This paper has been organised as follows: In section two, the (2,2)-supersymmetric sigma models with target spaces almost complex manifolds will be reviewed. In section three, the new (2,0)-supersymmetric sigma models with Nijenhuis symmetries, will be presented. In section four, examples of (2,2)- and (2,0)-supersymmetric sigma models with Nijenhuis symmetries and target spaces group manifolds will be given. In section five, the Poisson bracket algebra of the charges of (2,2)- and (2,0)-supersymmetric sigma models with Nijenhuis symmetries will be presented. In section six, the anomalies in the classical symmetries of the (2,0)-supersymmetric sigma model will be examined. In section seven, the massive extension of (2,0)- and (2,2)-supersymmetric sigma models with Nijenhuis symmetries and a topological twisting of the supersymmetry algebra with Nijenhuis charges will briefly be examined. A summary will be given in section eight.

2. The (2,2) model

Let $\mathcal{M}$ be a Riemannian manifold with metric g and a locally defined two-form b. The patching condition for b is $b' = b + dm$ where m is an one-form defined on the intersection of any two open sets of $\mathcal{M}$. The action of the (1,1)-supersymmetric model is

$$I = \int d^2x d\theta^+ d\theta^- (g+b)_{ij} D_-\phi^i D_+\phi^j \ , \tag{2.1}$$

where $(x^{\ddagger}, x^{=}, \theta^+, \theta^-)$ are the co-ordinates of (1,1) superspace, $\Xi^{(1,1)}$, $(x^{\ddagger}, x^{=}) = (x+t, t-x))$ are light-cone co-ordinates, the indices $i, j = 1, \ldots, \dim\ \mathcal{M}$, ϕ is

a (1,1)-superfield that takes values in $\mathcal{M}$, and D_-, D_+ are the supersymmetry derivatives of (1,1) superspace, *i.e.*

$$D_+^2 = i\partial_{\ddagger} \ , \qquad D_-^2 = i\partial_= \ . \tag{2.2}$$

The (2,0) and (0,2) supersymmetry transformations can be written in terms of (1,1) superfields as follows:

$$\begin{aligned} \delta_I \phi^i &= a_- I^i_j D_+ \phi^j \ , \\ \delta_J \phi^i &= a_+ J^i_j D_- \phi^j \ , \end{aligned} \tag{2.3}$$

where I and J are (1,1) tensors on the sigma model manifold $\mathcal{M}$ and a_+, a_- are the constant anti-commuting parameters of the transformations. These transformations leave the action (2.1) invariant provided that

$$\begin{aligned} \nabla^{(+)}_i I^j{}_k &= 0 \ , \qquad \nabla^{(-)}_i J^j{}_k = 0 \ , \\ g_{k(i} I^k{}_{j)} &= 0 \ , \qquad g_{k(i} J^k{}_{j)} = 0 \ , \end{aligned} \tag{2.4}$$

where

$$\Gamma^{(\pm)i}{}_{jk} = \{{}^{\,i}_{jk}\} \pm H^i{}_{jk} \ , \tag{2.5}$$

and

$$H_{ijk} = \frac{3}{2} \partial_{[i} b_{jk]} \ . \tag{2.6}$$

The algebra of transformations of eqn.(2.3) closes on-shell as follows:

$$\begin{aligned} [\delta_I, \delta'_I]\phi^i &= \delta_{N(I)}\phi^i + 2ia'_- a_- \partial_{\ddagger}\phi^i \ , \\ [\delta_J, \delta'_J]\phi^i &= \delta_{N(J)}\phi^i + 2ia'_+ a_+ \partial_=\phi^i \ , \\ [\delta_I, \delta_J]\phi^i &= 0 \ , \end{aligned} \tag{2.7}$$

provided that

$$I^2 = -1, \qquad J^2 = -1 \ , \tag{2.8}$$

where

$$\begin{aligned} \delta_{N(I)}\phi^i &\equiv a_{=}N(I)^i{}_{jk}D_+\phi^j D_+\phi^k \ , \\ \delta_{N(J)}\phi^i &\equiv a_{\ddagger}N(J)^i{}_{jk}D_-\phi^j D_-\phi^k \ , \end{aligned} \tag{2.9}$$

$a_{=}$, $a_{\ddagger}$ are the parameters of the transformations ($a_{=} = a'_- a_-$, $a_{\ddagger} = a'_+ a_+$ in the commutator (2.7)), and $N(I)$ and $N(J)$ are the Nijenhuis tensors of I and J, respectively. The Nijenhuis tensor of a (1,1) tensor I on $\mathcal{M}$ is

$$N(I)^i{}_{jk} = 2\big(I^m{}_{[j}\partial_{|m|}I^i{}_{k]} - I^i{}_m\partial_{[j}I^m{}_{k]}\big) \ . \tag{2.10}$$

The transformations (2.9) are symmetries of the action (2.1) because they appear in the commutator (2.7) of two symmetries of the theory together with the translations. Note that the translations are by themselves symmetries of the action (2.1). We can also verify this by a straightforward calculation using the following properties of the Nijenhuis tensor of the almost complex structures I and J:

$$\nabla^{(+)}_i N(I)_{jkl} = 0 \ , \qquad \nabla^{(-)}_i N(J)_{jkl} = 0 \ , \tag{2.11}$$

and

$$N(I)_{ijk} = N(I)_{[ijk]} \ , \qquad N(J)_{ijk} = N(J)_{[ijk]} \ . \tag{2.12}$$

The commutator of Nijenhuis transformations (2.9) with themselves, and with the (2,0) and (0,2) supersymmetry transformations given in (2.3) vanishes [10]. Note that

$$N(I)_{kij}I^k{}_m + (m,i) = 0, \qquad N(J)_{kij}J^k{}_m + (m,i) = 0 \ . \tag{2.13}$$

Finally, it is worth pointing out though that, in contradistinction to the case of (2,2)-supersymmetric sigma models with target spaces complex manifolds, the parameters of the transformations (2.7) cannot be promoted to semi-local ones because the algebra of supersymmetry transformations (2.3) and Nijenhuis symmetries (2.9) does not close.

3. The (2,0) models

Let $\mathcal{M}$ be a Riemannian manifold with metric g and a locally defined 2-form b as in the previous section, and $\mathcal{E}$ be a vector bundle over $\mathcal{M}$ with connection A ($\mathrm{rank}\,\mathcal{E} = \mathrm{k}$) and fibre metric h. We choose the connection A such that $\nabla_i h = 0^\star$. The action of (1,0)-supersymmetric sigma model is

$$I = -i \int d^2x d\theta^+ \big\{ (g_{ij} + b_{ij}) D_+ \phi^i \partial_= \phi^j + i h_{ab}\, \psi_-^a \nabla_+ \psi_-^b \big\} \ , \tag{3.1}$$

where $(x^{\ddagger}, x^{=}, \theta^+)$ are the co-ordinates of (1,0) superspace, $\Xi^{(1,0)}$, $(x^{\ddagger}, x^{=}) = (x+t, t-x)$ are light-cone co-ordinates, the indices $a, b = 1, \dots, k$, D_+ is the supersymmetry derivative ($D_+^2 = i\partial_{\ddagger}$) and

$$\nabla_+ \psi_-^b \equiv (D_+ \psi_-^b + D_+ \phi^i A_i{}^b{}_c \psi_-^c) \ . \tag{3.2}$$

The fields of (1,0) supersymmetric sigma model are the following: the scalar superfield $\phi(x, \theta^+)$ which is a map from the (1,0) superspace, $\Xi^{(1,0)}$, into the target manifold $\mathcal{M}$, and the spinor superfield $\psi_-^a(x, \theta^+)$ which is a section of the vector bundle $\xi_- \otimes \phi^* \mathcal{E}$ where ξ_- is the spin bundle over $\Xi^{(1,0)}$. The part of (3.1) that contains the ψ field is called either the fermionic or Yang-Mills sector of the sigma model action.

To find sigma models with (2,0) supersymmetry, we introduce the transformations

$$\begin{aligned} \delta_I \phi^i &= a_- I^i{}_j D_+ \phi^j \\ \delta_I \psi_-^a &= -A_i{}^a{}_b \delta_I \phi^i \psi_-^b + a_- \hat{I}^a{}_b \nabla_+ \psi_-^b \ . \end{aligned} \tag{3.3}$$

written in terms of (1,0) superfields, where I is a (1,1) tensor on $\mathcal{M}$, $\hat{I}$ is a (1,1) tensor on the fibre of $\mathcal{E}$ and a_- is the constant anti-commuting parameter of the transformations. The commutator of these transformations on the field ϕ is the

$\star$ Note that given a connection A of $\mathcal{E}$ with fibre metric h there always exist another connection of $\mathcal{E}$ with respect to which h is covariantly constant.

same as the one given in the previous section for the δ_I transformations on ϕ for the case (2,2)-supersymmetric sigma models, *i.e.* the commutator on ϕ of the transformations (3.3) closes to translations and Nijenhuis transformations provided that I is and almost complex structure ($I^2 = -1$). The commutator on the field ψ is

$$\begin{aligned}
[\delta_I, \delta'_I]\psi^a_- &= -A_i{}^a{}_b[\delta_I, \delta'_I]\phi^i\psi^b_- \\
&\quad -a'_-a_-(F_{kl}{}^a{}_b I^k{}_i I^l{}_j - F_{ij}{}^a{}_b)D_+\phi^i D_+\phi^j\psi^b_- \\
&\quad +2a'_-a_-(\nabla_j\hat{I}^a{}_b I^j{}_i - \hat{I}^a{}_c\nabla_i\hat{I}^c{}_b)D_+\phi^i\nabla_+\psi^b_- \\
&\quad +2ia'_-a_-\nabla_{\mp}\psi^a \ ,
\end{aligned} \tag{3.4}$$

where

$$F_{ij}{}^a{}_b = \partial_i A^a_{j\,b} - \partial_j A^a_{i\,b} + A^a_{i\,c}A^c_{j\,b} - A^a_{j\,c}A^c_{i\,b} \tag{3.5}$$

is the curvature of the connection A. There are two cases to consider the following: Case (i), the commutator on ψ closes *on-shell* to translations and to the Nijenhuis transformations

$$\begin{aligned}
\delta_N\phi^i &= a_{=}N^i_{jk}D_+\phi^j D_+\phi^k \ , \\
\delta_N\psi^a_- &= -A_i{}^a{}_b\delta_N\phi^i\psi^b_- \ ,
\end{aligned} \tag{3.6}$$

where $a_{=}$ is the parameter of the Nijenhuis transformations ($a_{=} = a'_-a_-$ in the commutator (3.4)), provided that

$$F_{kl}{}^a{}_b I^k{}_i I^l{}_j - F_{ij}{}^a{}_b = 0 \ . \tag{3.7}$$

Case (ii), the commutator (3.4) closes *off-shell* to translations and the Nijenhuis symmetry (3.6), provided that in addition to (3.7), the conditions

$$\hat{I}^2 = -1, \qquad \nabla_j\hat{I}^a{}_b I^j{}_i - \hat{I}^a{}_c\nabla_i\hat{I}^c{}_b = 0 \tag{3.8}$$

are satisfied. Therefore $\hat{I}$ is an almost complex structure on the fibre of $\mathcal{E}$. It is worth pointing out that given a connection A of $\mathcal{E}$, we can always choose another connection $\tilde{A} = A - \frac{1}{2}\hat{I}\nabla\hat{I}$ on $\mathcal{E}$ such that $\tilde{\nabla}_i\hat{I} = 0$ and $\tilde{\nabla}h = 0$, if $\nabla h = 0$, where h is the fibre metric of $\mathcal{E}$.

The commutator of the Nijenhuis (3.6) with the (2,0) supersymmetry transformations on both fields ϕ and ψ vanishes, *i.e.*

$$[\delta_I, \delta_N]\phi^i = 0, \qquad [\delta_I, \delta_N]\psi_-^a = 0 \ . \tag{3.9}$$

It is straightforward to verify this on the field ϕ because this commutator is the same as the one of the Nijenhuis with (2,0) supersymmetry transformations in the case of (2,2) models reviewed in section two. To examine the commutator of the Nijenhuis with the (2,0) supersymmetry transformations on the field ψ, there are two cases to consider the following: Case (i), the commutator on ψ vanishes *on-shell* provided that

$$F_{mn}{}^a{}_b I^m{}_{[i} N^n{}_{jk]} = 0 \ . \tag{3.10}$$

We can show that (3.10) is not an independent condition and it can be derived from the conditions that I is an almost complex structure and F satisfies (3.7) using the Bianchi identities. We can also show that

$$N^m{}_{[ij} F_{k]m}{}^a{}_b = 0 \ . \tag{3.11}$$

Case (ii), the commutator on ψ vanishes *off-shell* provided that, in addition to (3.10),

$$N(I)^i{}_{jk} \nabla_i \hat{I} = 0 \ . \tag{3.12}$$

The commutator of two Nijenhuis symmetries vanishes, as well, without further conditions.

The action is invariant under both (2,0) supersymmetry and Nijenhuis transformations provided that, in addition to the conditions obtained above for the closure of the algebra of these transformations, the following conditions are satisfied:

$$\nabla_i^{(+)} I^j{}_k = 0, \qquad g_{k(i} I^k{}_{j)} = 0 \qquad h_{c(a} \hat{I}^c{}_{b)} = 0 \ . \tag{3.13}$$

To summarise, the independent conditions for the invariance of the action and the *on-shell* closure of the algebra of supersymmetry and Nijenhuis transformations

of (2,0)-supersymmetric sigma models are the following: the first equation in (2.8), (3.7) and (3.13). Note that in this case we can set $\hat{I} = 0$. The independent conditions for the invariance of the action and the *off-shell* closure of the algebra of supersymmetry and Nijenhuis transformations are the following: the first equation in (2.8), (3.7), (3.8), (3.12) and (3.13). Therefore, the target manifolds of sigma models with *on-shell* (2,0) supersymmetry and Nijenhuis symmetry are almost complex manifolds, the almost complex structure I is covariantly constant with respect to the $\Gamma^{(+)}$ connection, i.e. the holonomy of $\Gamma^{(+)}$ is a subgroup of $U(m)$ ($\dim\mathcal{M} = 2$m), the metric g is hermitian with respect to I and the curvature of the connection A of the bundle $\mathcal{E}$ is an (1,1) form with respect to I. In addition to the above restrictions on the geometry of the target manifold $\mathcal{M}$ and vector bundle $\mathcal{E}$ required by on-shell (2,0) supersymmetry, the bundle $\mathcal{E}$ of sigma models with *off-shell* (2,0) supersymmetry and Nijenhuis symmetry must admit a fibre almost complex structure $\hat{I}$, i.e. the structure group of $\mathcal{E}$ is a subgroup of $U(\frac{k}{2})$, and the fibre metric h is hermitian with respect to $\hat{I}$.

4. Models on group manifolds

Some explicit examples of models with (2,2) or (2,0) supersymmetry and Nijenhuis symmetry are found by consideration of sigma models with a group manifold as their target space. Let K be a group manifold with Lie algebra $\mathcal{L}(K)$. The left L^A and right R^A invariant frames on K are defined as follows:

$$k^{-1}dk = L^A t_A \qquad dk\, k^{-1} = R^A t_A, \qquad k \in K \ , \tag{4.1}$$

where $\{t_A\}$ is a basis in $\mathcal{L}(K)$, $[t_A, t_B] = f_{AB}{}^C t_C$, the indices $A, B, C = 1, \dots, \dim\mathcal{L}(\mathrm{K})$, and $f_{AB}{}^C$ are the structure constants of $\mathcal{L}(K)$. The Maurer-Cartan equations are

$$dL^A = -\frac{1}{2} f_{BC}{}^A L^B L^C \qquad dR^A = \frac{1}{2} f_{BC}{}^A R^B R^C \ . \tag{4.2}$$

The sigma model metric g and torsion H are chosen to be the bi-invariant tensors

$$\begin{aligned} g_{ij} &= \kappa_{AB} L_i^A L_j^B = \kappa_{AB} R_i^A R_j^B \ , \\ H_{ijk} &= -\frac{\lambda}{2} f_{ABC} L_i^A L_j^B L_k^C = -\frac{\lambda}{2} f_{ABC} R_i^A R_j^B R_k^C \ , \end{aligned} \tag{4.3}$$

where κ_{AB} is an invariant non-degenerate quadratic form on $\mathcal{L}(K)$, $f_{ABC} = f_{AB}{}^D \kappa_{DC}$ and λ is a real number. The Wess-Zumino-Witten models that we will consider here are those for which $|\lambda| = 1$. For these values of λ both $\Gamma^{(+)}$ and $\Gamma^{(-)}$ connections are flat and the group manifold is parallelizable with respect to both connections. In the following we will choose $\lambda = 1$.

We will first consider the sigma models with (2,2) supersymmetry (see also ref. [9]). The task is to find solutions to the conditions required by (2,2)-supersymmetry on the almost complex structures I and J, *i.e.* to find solutions to the conditions of eqns. (2.4). We can solve the first two equations in (2.4) by setting

$$I^i{}_j = L^i_A I^A{}_B L^B{}_j \ , \qquad J^i{}_j = R^i_A J^A{}_B R^B{}_j \ , \tag{4.4}$$

where the matrices $\{I^A{}_B\}$ and $\{J^A{}_B\}$ are constant*. The condition that both $I^i{}_j$ and $J^i{}_j$ are almost complex structures implies that

$$I^A{}_C I^C{}_B = -\delta^A{}_B \qquad J^A{}_C J^C{}_B = -\delta^A{}_B \tag{4.5}$$

The last two equations of (2.4) then become

$$\kappa_{CD} I^C{}_A I^D{}_B = \kappa_{AB} \qquad \kappa_{CD} J^C{}_A J^D{}_B = \kappa_{AB} \ . \tag{4.6}$$

The conditions for existence of (2,2) supersymmetric sigma models reduces in this case to the algebraic equations (4.5) and (4.6). Since one may set $I^A{}_B = J^A{}_B$, a

$\star$ We use the same symbols I and J to denote the almost complex structure on K and their associated constant tensors on $\mathcal{L}(K)$. To avoid confusion when we refer to the latter, we will use Lie algebra indices.

group manifold K is the target manifold of a (2,2)-supersymmetric sigma model with Nijenhuis symmetries provided that there is a (constant) complex structure $I^A{}_B$ and an invariant quadratic form κ on the Lie algebra $\mathcal{L}(K)$ which is Hermitian with respect to $I^A{}_B$. If K is a simple compact Lie group, then the space of independent parameters that parameterise classically the different (2,2)-supersymmetric sigma models with Nijenhuis symmetry (the moduli space of the theory) is as follows: First the constant positive conformal factor that scales quadratic form κ. This space is topologically $\mathcal{R}^+$ and parameterises the size of K. Note that every simple Lie group has a unique invariant non-degenerate quadratic form up to scaling with a constant conformal factor. Second, for each metric there are $\frac{SO(2m)}{U(m)}$ complex structures on $\mathcal{L}(K)$ that satisfy (4.6)($\dim K = 2m$). So the moduli space of a (2,2)-supersymmetric sigma model with Nijenhuis symmetries and target space an even-dimensional group manifold can be thought as a bundle with base space $\mathcal{R}^+$ and fibre the space $\frac{SO(2m)}{U(m)} \times \frac{SO(2m)}{U(m)}$. Quantum mechanically though the coupling constant, *i.e* the conformal factor, of the WZW model is quantised and therefore the space $\mathcal{R}^+$ becomes a lattice. The above results can be easily generalised for any semisimple group K.

To give examples of sigma models with on-shell (2,0) supersymmetry and target space group manifolds, we must find solutions to the equation (3.7) in addition to those satisfied by the almost complex structure I. The conditions on the almost complex structure I required by (2,0) supersymmetry are the same as those required by (2,2) supersymmetry and therefore they reduce to the algebraic conditions (4.5) and (4.6) for $I^A{}_B$ and κ_{AB}. For compact simple Lie groups the space of parameters is again a bundle space with base space $\mathcal{R}^+$ and fibre $\frac{SO(2m)}{U(m)}$. Now it remains to solve the equation (3.7). For this we will assume further that the fermionic sector of the (2,0) sigma model is invariant under the left action of K† If this is the case the bundle $\mathcal{E}$ is topologically trivial and we can use the standard trivialisation of $\mathcal{E}$ to identify the connections of $\mathcal{E}$ with the $\mathcal{L}(H)$-valued one-forms

† For more details on the symmetries of the fermionic sector of the sigma model action see ref.[11].

on K where H is the gauge group of the connections A. The left invariant connections can now be written as

$$A_i = L_i^A \omega_A \tag{4.7}$$

where ω is constant and can be thought as a linear map from the Lie algebra $\mathcal{L}(K)$ of K into $\mathcal{L}(H)$. The equation (3.7) is then equivalent to

$$(-f_{AB}^E \omega_E^r + f_{st}^r \omega_A^s \omega_B^t) I^A{}_C I^B{}_D = (-f_{CD}^E \omega_E^r + f_{st}^r \omega_C^s \omega_D^t) \ , \tag{4.8}$$

where f_{st}^r are the structure constants $\mathcal{L}(H)$ and $r, s, t = 1, \ldots, \dim \mathcal{L}(H)$. Using the complex structure $I^A{}_B$ to decompose $\mathcal{L}(K) \otimes \mathcal{C}$ into holomorphic $\mathcal{L}(K)^{(1,0)}$ and anti-holomorphic $\mathcal{L}(K)^{(0,1)}$ subspaces, we can rewrite the equation (4.8) as follows:

$$-f_{\alpha\beta}^E \omega_E^r + f_{st}^r \omega_\alpha^s \omega_\beta^t = 0 \ , \qquad -f_{\bar{\alpha}\bar{\beta}}^E \omega_E^r + f_{st}^r \omega_{\bar{\alpha}}^s \omega_{\bar{\beta}}^t = 0 \ , \tag{4.9}$$

where the indices $A = (\alpha, \bar{\alpha})$ and $\alpha, \beta = 1, \cdots, m$. Further simplification of (4.9) does not seem possible for the case that I is an almost complex structure. Observe though that $\omega = 0$ is a solution of (4.9). However if the Nijenhuis tensor of I is zero, one can show that $f_{\alpha\beta\gamma} = 0$ which in turn implies that $\mathcal{L}(K)^{(1,0)}$ and $\mathcal{L}(K)^{(0,1)}$ are subalgebras of $\mathcal{L}(K) \otimes \mathcal{C}$. The equation (4.9) then implies that ω_α^s is a Lie algebra homomorphism from $\mathcal{L}(K)^{(1,0)}$ into $\mathcal{L}(H) \otimes \mathcal{C}$. Finally to find (2,0) sigma models with off-shell supersymmetry and target space group manifolds, we assume that the connection A satisfies (in addition to the requirements necessary for the existence of (2,0) models with on-shell supersymmetry) the condition $\nabla \hat{I} = 0$ which implies all the remainning conditions for off-shell closure of the algebra of (2,0) supersymmetry and Nijenhuis transformations. The condition $\nabla \hat{I} = 0$ has solutions provided the representation of $\mathcal{L}(H)$ on the fields ψ has an invariant (constant) complex structure and the fibre metric h is hermitian with respect to $\hat{I}$.

5. The Poisson bracket algebra of charges

The conserved currents of the (2,0) supersymmetric sigma model are the energy momentum tensor, the (1,0) and (2,0) supersymmetry currents and a current corresponding to a $U(1)$ charge that rotates the two supersymmetry charges. These currents expressed in terms of (1,0) superfields are the following: The current

$$\mathcal{G}^{0}_{\ddagger +} = g_{ij} D_+\phi^i \partial_{\ddagger}\phi^j - \frac{i}{3} H_{ijk} D_+\phi^i D_+\phi^j D_+\phi^k \tag{5.1}$$

that has components the (1,0) supersymmetry current $\mathcal{S}^{0}_{\ddagger +} = \mathcal{G}^{0}_{\ddagger +}|$ and the $\mathcal{T}_{\ddagger\ddagger}$ component of the energy momentum tensor, $\mathcal{T}_{\ddagger\ddagger} = -i\frac{1}{2}\big(D_+\mathcal{G}^{0}_{\ddagger +}\big)|$, the current

$$\mathcal{G}^{1}_{++} = -I_{ij} D_+\phi^i D_+\phi^j \ , \tag{5.2}$$

that has components the U(1) current $\mathcal{J}_{\ddagger} = \mathcal{G}^{1}_{++}|$ and the (2,0) supersymmetry current $\mathcal{S}^{1}_{\ddagger +} = i\frac{1}{2} D_+\mathcal{G}^{1}_{\ddagger +}|$ and the current

$$\mathcal{P}_{+\ddagger} = \frac{2}{3} N_{ijk} D_+\phi^i D_+\phi^j D_+\phi^k \tag{5.3}$$

that has components the Nijenhuis currents $\mathcal{N}_{+\ddagger} = (\mathcal{P}_{+\ddagger})|$ and $\mathcal{N}_{\ddagger\ddagger} = -\frac{1}{4}(D_+\mathcal{P}_{+\ddagger})|$, where the vertical line denotes the evaluation of the corresponding expression at $\theta^+ = 0$. All the above currents are chiral, i.e. $\partial_{=}\mathcal{G}^{0}_{+\ddagger} = 0$, $\partial_{=}\mathcal{G}^{1}_{\ddagger} = 0$ and $\partial_{=}\mathcal{P}_{+\ddagger} = 0$.

The most convenient way to present the Poisson bracket algebra of the charges of the above currents is to 'smear' them with the parameters of the associated transformations. The charges are then the following:

$$S^{0}(\epsilon_{=}) = \int dx\, d\theta^+\, \epsilon_{=}\, \mathcal{G}^{0}_{\ddagger +} \ , \tag{5.4}$$

$$S^{1}(a_{-}) = \int dx\, d\theta^+\, a_{-}\, \mathcal{G}^{1}_{\ddagger} \ , \tag{5.5}$$

and

$$N(a_=) = \int dx d\theta^+ \, a_= \, \mathcal{P}_{\ddagger +} \ . \tag{5.6}$$

It is worth pointing out that only certain linear combinations of all the possible charges of the model enter in the above expressions. In particular, $S^{0}(\epsilon_=)$ is a linear combination of the charges $T_\ddagger \equiv E + P$ and (1,0) supersymmetry charge as one can easily verify by observing that

$$S^{0}(\epsilon_=) = (D_+\epsilon_=)| \int \ dx \ \overset{0}{\mathcal{S}}_{\ddagger +} + 2i\epsilon_=| \int \ dx \ \mathcal{T}_{\ddagger \ddagger} \ , \tag{5.7}$$

and $\epsilon_= = \epsilon_=(\theta^+)$, where $\mathcal{T}_{\ddagger\ddagger}(\equiv -i\frac{1}{2}D_+\mathcal{S}^0{}_{+\ddagger}|)$ is the indicated component of the energy momentum of the theory. Similarly, $S^1(a_-)$ is proposional to (2,0) supersymmetry charge and $N(a_=)$ is proposional to the charge of the $\mathcal{N}_{\ddagger\ddagger}$ Nijenhuis current since both parameters a_- and $a_=$ are constant, *i.e.* independent of the coordinates of the (1,0) superspace. These charges can be easily expressed in terms of the component fields $\phi = \phi|$ and $\lambda_+ = (D_+\phi)|$ of the theory by performing the integration over the odd variable θ and then substitute in the resulting expression the component fields.

The non-vanishing Poisson brackets of the charges (5.4)-(5.6) of the (2,0) supersymmetric sigma model with Nijenhuis symmetries are the following:

$$\{S^{0}(\epsilon_=), S^{0}(\epsilon'_=)\} = S^{0}(iD_+\epsilon_= D_+\epsilon'_=), \quad \{S^1(a_-), S^1(a'_-)\} = -4iS^{0}(a_-a'_-) - 2N(a_-a'_-) \ . \tag{5.8}$$

This Poisson bracket algebra of the charges of the (2,0)-supersymmetric sigma model with Nijenhuis symmetries is not isomorphic to the standard (2,0) supersymmetry algebra. Indeed to compare the (2,0) supersymmetry algebra (5.8) and the corresponding (2,0) supersymmetry algebra (1.1), let us set

$$S^{0}(\epsilon_=) \equiv (D_+\epsilon_=)|\overset{0}{S}_+ + 2i\epsilon_=|T_\ddagger, \qquad S^1(a_-) \equiv 2ia_-S^1_+, \qquad N(a_=) \equiv -4a_= N_\ddagger \tag{5.9}$$

The Poisson bracket algebra (5.8) then becomes

$$\{\overset{0}{S}_+, \overset{0}{S}_+\} = 2T_{\mp} , \qquad \{S^1_+, S^1_+\} = 2(T_{\mp} + N_{\mp}) ; \tag{5.10}$$

the remaining Poisson brackets vanish. Next we define new generators as follows:

$$\tilde{\overset{0}{S}}_+ = \overset{0}{S}_+ + S^1_+ , \qquad \tilde{S}^1_+ = -\overset{0}{S}_+ + S^1_+ , \qquad \tilde{T}_{\mp} = 2T_{\mp} + N_{\mp} . \tag{5.11}$$

In terms of these new charges, the algebra (5.10) can be rewritten as

$$\{\tilde{\overset{0}{S}}_+, \tilde{\overset{0}{S}}_+\} = 2\tilde{T}_{\mp} , \qquad \{\tilde{S}^1_+, \tilde{S}^1_+\} = 2\tilde{T}_{\mp} , \qquad \{\tilde{\overset{0}{S}}_+, \tilde{S}^1_+\} = 2N_{\mp} . \tag{5.12}$$

The algebra (5.8) rewritten as (5.12) is not isomorphic to the corresponding (2,0) supersymmetry algebra (1.1) because the Poisson bracket of first supersymmetry charge $\tilde{\overset{0}{S}}_+$ with the second $\tilde{S}^1_+$ does not vanish as in (1.1) but rather it gives a new (central) charge $N_{\mp}$ of the algebra which has Lorentz weight one. Another feature of the algebra (5.12) is that the $SO(2)$ rotation that rotates the supersymmetry charges $\tilde{\overset{0}{S}}_+$ and $\tilde{S}^1_+$ to each other and leaves the rest of the charges invariant is *not* an automorphism of the algebra. However the algebra (5.12) has an $SO(1,1)$ (non-compact) automorphism R that acts on its generators as follows:

$$\begin{aligned} \{R, \tilde{\overset{0}{S}}_+\} &= \tilde{S}^1_+ , \qquad \{R, \tilde{S}^1_+\} = \tilde{\overset{0}{S}}_+ , \\ \{R, T_{\mp}\} &= 2N_{\mp} , \qquad \{R, N_{\mp}\} = 2T_{\mp} . \end{aligned} \tag{5.13}$$

This automorphism of the algebra is not realised by a transformation on the fields of the (2,0)-supersymmetric sigma model studied in section three.

To compute the Poisson bracket algebra of charges of (2,2)-supersymmetric sigma models with Nijenhuis symmetries, section 2, it is convenient to express all

the charges in terms of (1,1) superfields. The currents of the theory are

$$
\begin{aligned}
\overset{0}{\mathcal{G}}_{+\!\!\!+} &= g_{ij} D_+ \phi^i \partial_{+\!\!\!+} \phi^j - \frac{i}{3} H_{ijk} D_+ \phi^i D_+ \phi^j D_+ \phi^k \ , \\
\overset{0}{\mathcal{G}}_{-=} &= g_{ij} D_- \phi^i \partial_{=} \phi^j - \frac{i}{3} H_{ijk} D_- \phi^i D_- \phi^j D_- \phi^k \ , \\
\mathcal{G}^1_{++} &= -I_{ij} D_+ \phi^i D_+ \phi^j \ , \\
\mathcal{G}^1_{--} &= -J_{ij} D_- \phi^i D_- \phi^j \ , \\
\mathcal{P}_{+\!\!\!+} &= \frac{2}{3} N_{ijk} D_+ \phi^i D_+ \phi^j D_+ \phi^k \ , \\
\mathcal{P}_{-=} &= \frac{2}{3} N_{ijk} D_- \phi^i D_- \phi^j D_- \phi^k \ ,
\end{aligned}
\tag{5.14}
$$

where the components of the $\overset{0}{\mathcal{G}}_{+\!\!\!+}$, $\overset{0}{\mathcal{G}}_{-=}$ are the energy momentum tensor and the (1,0) and (0,1) supersymmetry currents, the components of $\mathcal{G}^1_{++}$ and $\mathcal{G}^1_{--}$ are the (2,0) and (0,2) supersymmetry currents and two $U(1)$ currents, and the components of $\mathcal{P}_{+\!\!\!+}$ and $\mathcal{P}_{-=}$ are the currents corresponding to Nijenhuis symmetries. All the above currents are conserved, *i.e.* $D_- \overset{0}{\mathcal{G}}_{++} = 0$, $D_+ \overset{0}{\mathcal{G}}_{-=} = 0$, $D_- \mathcal{G}^1_{++} = 0$, $D_+ \mathcal{G}^1_{--} = 0$, $D_- \mathcal{P}_{+\!\!\!+} = 0$ and $D_+ \mathcal{P}_{-=} = 0$. The associated 'smeared' charges are as follows:

$$
\begin{aligned}
\overset{0}{S}(\epsilon_{=}) &= \int dx\, d\theta^+\, \epsilon_{=}\, \overset{0}{\mathcal{G}}_{+\!\!\!+} \ , \qquad & \overset{0}{S}(\epsilon_{+\!\!\!+}) &= \int dx\, d\theta^-\, \epsilon_{+\!\!\!+}\, \overset{0}{\mathcal{G}}_{-=} \ , \\
S^1(a_-) &= \int dx\, d\theta^+\, a_-\, \mathcal{G}^1_{++} \ , \qquad & S^1(a_+) &= \int dx\, d\theta^-\, a_+\, \mathcal{G}^1_{--} \ , \\
N(a_{=}) &= \int dx\, d\theta^+\, a_{=}\, \mathcal{P}_{+\!\!\!+} \ , \qquad & N(a_{+\!\!\!+}) &= \int dx\, d\theta^-\, a_{+\!\!\!+}\, \mathcal{P}_{-=} \ .
\end{aligned}
\tag{5.15}
$$

The non-vanishing Poisson brackets of the charges (5.15) of the (2,2)-supersymmetric sigma model are as follows:

$$
\begin{aligned}
\{\overset{0}{S}(\epsilon_{=}), \overset{0}{S}(\epsilon'_{=})\} &= \overset{0}{S}(iD_+\epsilon_{=} D_+\epsilon'_{=}), \quad & \{S^1(a_-), S^1(a'_-)\} &= -4i\overset{0}{S}(a_- a'_-) - 2N(a_- a'_-), \\
\{\overset{0}{S}(\epsilon_{+\!\!\!+}), \overset{0}{S}(\epsilon'_{+\!\!\!+})\} &= \overset{0}{S}(iD_-\epsilon_{+\!\!\!+} D_-\epsilon'_{+\!\!\!+}), \quad & \{S^1(a_+), S^1(a'_+)\} &= -4i\overset{0}{S}(a_+ a'_+) - 2N(a_+ a'_+) \ .
\end{aligned}
\tag{5.16}
$$

The algebra of charges (5.16) of the (2,2)-supersymmetric sigma model is two commuting copies of the algebra (5.8) of the (2,0) model. Using arguments similar to

those for the (2,0) case, we can show that (5.16) is not isomorphic to the corresponding (2,2) supersymmetry algebra (1.1).

6. Anomalies

Generic sigma models with (2,0) supersymmetry have a different number of left- from right- chiral fermions and therefore some of their symmetries are quantum mechanically anomalous [12, 13]. To examine the anomalies for (2,0)-supersymmetric sigma models with Nijenhuis symmetries, we quantise the theory in the background field method. As in the case of (2,0)-supersymmetric sigma models with target spaces complex manifolds, the model can be quantised in such a way that the background/quantum field split symmetry [14], (1,0) supersymmetry and sigma model manifold reparameterisations are manifestly preserved quantum mechanically. The arguments for this are similar to those of ref. [13] and they will not be repeated here. The transformations that may be anomalous quantum mechanically are the following: The frame rotations of the tangent bundle of the sigma model manifold

$$\delta_U \omega_i^{(-)A}{}_B = -\partial_i U^A{}_B + U^A{}_C \omega_i^{(-)C}{}_B - \omega_i^{(-)A}{}_C U^C{}_B \ , \tag{6.1}$$

where $\omega^{(-)}$ is a spin connection of $\Gamma^{(-)}$ and U is the infinitesimal gauge parameter, the gauge transformations of the connection A

$$\delta_L A_i{}^a{}_b = -\partial_i L^a{}_b + L^a{}_c A_i{}^c{}_b - A_i{}^a{}_c L^c{}_b \ , \tag{6.2}$$

where L is the infinitesimal parameter, the (2,0) supersymmetry transformations

$$\delta_I \phi^i = a_- I^i{}_j D_+ \phi^j \ , \tag{6.3}$$

and the Nijenhuis symmetries

$$\delta_N \phi^i = a_= N^i{}_{jk} D_+ \phi^j D_+ \phi^k \ , \tag{6.4}$$

where N is the Nijenhuis tensor of the almost complex structure I. The transformations (6.1) and (6.2) are invariances of the sigma model action, *i.e.* transformations

of the fields that leave invariant the classical action provided that they are compensated by appropriate transformations of the couplings*, and no Noether currents are associated with them. Because the (1,0) supersymmetry is manifestly preserved in the quantum theory one can use (1,0) superfields to study the anomalies of the classical symmetries (6.1)-(6.4).

The anomalies in the frame rotations of the sigma model tangent bundle can be derived from the familiar descent equations [15]

$$P_4 = dQ^0_3\ , \qquad \delta_U Q^0_3 + dQ^1_2 = 0\ , \qquad \delta_U Q^1_2 + dQ^2_1 = 0\ , \qquad \delta_U Q^2_1 + dQ^3_0 = 0\ , \tag{6.5}$$

where P_4 is the first Pontrjagin class of the tangent bundle of the sigma model target space, $P_4 = \mathrm{tr}R^2$, and U is the infinitesimal parameter of the transformations. Similar descent equations can be used to compute the anomalies of the gauge transformations of the connection A. The anomalies in the frame rotations of the sigma model tangent bundle [7] are

$$\Delta(U) = iy \int d^2x d\theta^+\, Q^1_2\big(U, \omega^{(-)}\big)_{ij} D_+\phi^i \partial_= \phi^j\ , \tag{6.6}$$

where

$$Q^1_2(U, \omega^{(-)}) = U^A{}_B(\phi) d\omega^{(-)B}{}_A\ , \tag{6.7}$$

and $y = \frac{\hbar}{2\pi}$ is a numerical coefficient computed in perturbation theory [16], and similarly the anomalies in the gauge transformations of the connection A are

$$\Delta(L) = -iy \int d^2x d\theta^+\, Q^1_2(L, A)_{ij} D_+\phi^i \partial_= \phi^j\ , \tag{6.8}$$

where L is the infinitesimal parameter of the gauge transformations. The connections $\omega^{(-)}$ and A that enter in the expressions for the anomalies are not uniquely

$\star$ Note that the equations (6.1) and (6.2) denote only the transformations induced on the couplings by these invariances.

specified by the descent equations. In fact the anomalies (6.6) and (6.8) can be expressed in terms of any connection of the tangent bundle of $\mathcal{M}$ and $\mathcal{E}$, respectively, by adding appropriate finite local counterterms in the effective action. However as we will see below in the supersymmetric case it is convenient to express the anomalies (6.6) and (6.8) as above. The rest of the anomalies are specified by consistency conditions. One can derive these consistency conditions by applying the commutator of two symmetries on the effective action of the quantum theory of a model and then define the non-vanishing variations of the effective action as the corresponding anomalies. The consistency conditions for the (2,0)-supersymmetric sigma model with Nijenhuis symmetries are the following:

$$\delta_I \Delta(U) - \delta_U \Delta_I(a_-) = 0 \ , \qquad \delta_I \Delta(L) - \delta_L \Delta_I(a_-) = 0 \ , \tag{6.9}$$

$$\begin{aligned} \delta_I \Delta_N(a_=) - \delta_N \Delta_I(a_-) &= 0 \ , \\ \delta_N \Delta(U) - \delta_U \Delta_N(a_=) &= 0 \ , \\ \delta_N \Delta(L) - \delta_L \Delta_N(a_=) &= 0 \ , \\ \delta_N \Delta_N(a'_=) - \delta'_N \Delta_N(a_=) &= 0 \ , \end{aligned} \tag{6.10}$$

and

$$\delta_I \Delta_I(a'_-) - \delta'_I \Delta_I(a_-) = \Delta_N(a_- a'_-) \ , \tag{6.11}$$

where Δ_I is the (2,0) supersymmetry anomaly and Δ_N is the anomaly of the Nijenhuis symmetry. Solving (6.9) for the (2,0) supersymmetry anomaly Δ_I, we get

$$\Delta_I(a_-) = 3y \int d^2x d\theta^+ \left(Q_3^0(\omega^{(-)}, A)\right)_{ijk} \delta_I \phi^i D_+ \phi^j \partial_= \phi^k \ , \tag{6.12}$$

where

$$Q_3^0(\omega^{(-)}) = \mathrm{tr}[\omega^{(-)} d\omega^{(-)} + \frac{2}{3}\left(\omega^{(-)}\right)^3] \tag{6.13}$$

is the Chern-Simons form of the connection $\omega^{(-)}$ and similarly for the Chern-Simons

form $Q_3^0(A)$ for the connection A, and

$$Q_3^0(\omega^{(-)}, A) = Q_3^0(\omega^{(-)}) - Q_3^0(A) \ . \tag{6.14}$$

In fact (6.9) specifies the (2,0) supersymmetry anomaly up to a term invariant under both frame rotations and gauge transformations. A direct computation of the one-loop effective action reveals that such a term does not appear. From the consistency condition (6.11), we get that the anomaly of the Nijenhuis symmetry is the following:

$$\Delta_N(a_=) = 3y \int d^2x d\theta^+ \left(Q_3^0(\omega^{(-)}, A)\right)_{ijk} \delta_N \phi^i D_+ \phi^j \partial_= \phi^k \ . \tag{6.15}$$

To prove this we have used that the curvature of the connections $\omega^{(-)}$ and A are (1,1) forms with respect to the almost complex structure I. The rest of the consistency conditions (6.9) and (6.10) are also satisfied due to eqns. (3.10) and (3.11) of section three and similar equations satisfied by the curvature $R^{(-)}$ of the $\omega^{(-)}$ connection. The latter follows from the equations (2.11)-(2.13) of section two that involve the Nijenhuis tensor of the almost complex structure I and $R^{(-)}_{ijkl} = R^{(+)}_{klij}$.

To discuss a possible cancellation of the anomalies (6.6), (6.8), (6.12) and (6.15), we will briefly review the case that $\mathcal{M}$ is a complex manifold and $\mathcal{E}$ is a holomorphic vector bundle over $\mathcal{M}$ [13], *i.e.* there are not Nijenhuis symmetries in the theory. In this case, one can introduce complex co-ordinates on $\mathcal{M}$ and an exterior derivation

$$d_I = i(\partial - \bar{\partial}) \tag{6.16}$$

where ∂ and $\bar{\partial}$ are the exterior derivatives with respect the holomorphic and anti-holomorphic co-ordinates of $\mathcal{M}$, respectively. The exterior derivation d_I is associated with the complex structure I and satisfies the following conditions:

$$d_I^2 = 0 \ , \qquad dd_I + d_I d = 0 \ . \tag{6.17}$$

Using (6.17), the Poincaré lemma, the Dolbeault-Grothendieck lemma and the fact

that

$$P_4(\omega^{(-)}, A) \equiv P_4(\omega^{(-)}) - P_4(A) \tag{6.18}$$

is a (2,2)-form on $\mathcal{M}$ with respect to the complex structure I, one can write the three-form $Q^0_3(\omega^{(-)}, A)$ as follows:

$$Q^0_3(\omega^{(-)}, A) = dX + d_I Y \ , \tag{6.19}$$

where X is a two-form and Y is a (1,1)-form with respect to I on $\mathcal{M}$. Next the finite local counterterm

$$\Gamma_{fl} = -iy \int dx\, d\theta^+ \, (X_{ij} - Y_{ik} I^k{}_j) D_+ \phi^i \partial_= \phi^j \tag{6.20}$$

cancels the anomalies (6.6), (6.8) and (6.12) [13]. However the finite local counterterm (6.20) depends explicitly on the holomorphic structure of $\mathcal{M}$ and $\mathcal{E}$ and therefore induces new anomalies in the holomorphic gauge transformations of the tangent bundle of $\mathcal{M}$ and $\mathcal{E}$. The holomorphic anomalies are cancelled by anomalous variations of the metric and Wess-Zumino term of the theory. Finally, there is global anomaly in the theory which cancels provided that the four-form (6.18) is exact [17].

Now we will turn our attention to the case that the almost complex structure I is not integrable. It is well known that for every vector valued form on $\mathcal{M}$ one can introduce an exterior derivation. In particular there are exterior derivatives d_I and d_N associated with the almost complex structure I and the Nijenhuis tensor N of I, respectively (see for example ref.[10]). One can show that the derivations d, d_I and d_N obey the following algebra:

$$\begin{aligned} d^2 =& 0 \ , \qquad 2d_I^2 = d_N \ , \qquad dd_I + d_I d = 0 \ , \\ dd_N &- d_N d = 0 \ , \qquad d_I d_N - d_N d_I = 0 \ . \end{aligned} \tag{6.21}$$

Therefore in this case the derivation d_I is not nilpotent and there is no analogue of the Dolbeault-Grothendieck lemma that it is necessary to write $Q^0_3(\omega^{(-)}, A)$ as in

(6.19). So for generic sigma models with target spaces almost complex manifolds, the (2,0) supersymmetry and Nijenhuis transformations are anomalous quantum mechanically. However the anomalies due to frame rotations of the tangent bundle of $\mathcal{M}$ and the gauge transformations of A still cancel by anomalous variation of the Wess-Zumino term of the theory as in the case of (1,0)-supersymmetric sigma models in refs. [7, 16]. In some special cases of sigma models with Nijenhuis symmetries the (2,0) supersymmetry and Nijenhuis anomalies cancel as well. Indeed if the four-form $P_4(\omega^{(-)}, A)$ of eqn. (6.18) can be chosen to be zero, then $Q^0_3(\omega^{(-)}, A) = dZ$ and the finite local counterterm

$$\Gamma_{fl} = -iy \int dx\, d\theta^+ \, Z_{ij} D_+\phi^i \partial_= \phi^j \tag{6.22}$$

can be added in the effective action of the theory such that the anomalies in the (2,0) supersymmetry and Nijenhuis transformations vanish. Examples of such (2,0)-supersymmetric sigma models are those studied in section four with target spaces group manifolds. For these models $\omega^{(-)} = 0$ and one can choose $A = 0$ so all the anomalies (6.6), (6.8), (6.12) and (6.15) vanish identically.

Another example is the anomaly cancellation in the case of (2,2)-supersymmetric sigma model. This model is not expected to be anomalous because it has equal number of left and right handed fermions. One of the conditions for a (2,0)-supersymmetric sigma model to be (2,2)-supersymmetric is to set $A = \Gamma^{(-)}$ in which case $P_4(\omega^{(-)}, A) = 0$ and this condition is sufficient for the cancellation of (2,0) supersymmetry (6.12) and Nijenhuis (6.15) anomalies. A similar argument can be used to prove that the (0,2) supersymmetry anomaly and associated Nijenhuis one cancel as well.

7. Concluding Remarks

7.1. Massive (2,0) models

The results of section three on massless (2,0)-supersymmetric sigma models with target spaces almost complex manifolds can be extended to massive ones. For this, the action of massive sigma models with (1,0) supersymmetry [18, 19] is the following:

$$I_m = -i \int dx\, d\theta^+ \{(g+b)_{ij} D_+ \phi^i \partial_= \phi^j + i h_{ab} \phi^a_- \nabla_+ \psi^b_- + i m s_a(\phi) \psi^a_-\} \tag{7.1}$$

where the fields ϕ, ψ and the couplings g, b, A are defined as in section three. The only new coupling is s_a which can be thought as a section of the bundle $\mathcal{E}$ over $\mathcal{M}$ and m is a mass parameter. The action (7.1) is manifestly (1,0) supersymmetric. The (2,0) supersymmetry transformations written in (1,0) superfields are

$$\begin{aligned} \delta_I \phi^i &= a_- I^i{}_j D_+ \phi^j \ , \\ \delta_I \psi^a &= -i a_- \hat{I}^a{}_b \mathcal{S}^b + \frac{1}{2} m a_- t^a(\phi) \end{aligned} \tag{7.2}$$

where I is a (1,1) tensor of $\mathcal{M}$, $\hat{I}$ is a tensor on the fibre of $\mathcal{E}$, t^a is a section of $\mathcal{E}$ and

$$\mathcal{S}^a \equiv 2i \nabla_+ \psi^a_- + i m s^a \tag{7.3}$$

is the field equation for ψ. The transformations (7.2) leave the action (7.1) invariant and close on-shell to translations and Nijenhuis transformations,

$$\begin{aligned} \delta_N \phi^i &= a_= N^i_{jk} D_+ \phi^j D_+ \phi^k \ , \\ \delta_N \psi^a_- &= -A_i{}^a{}_b \delta_N \phi^i \psi^b_- \ , \end{aligned} \tag{7.4}$$

provided that, in addition to the conditions described in section three for the invariance of the action of the massless (2,0) model ($m = 0$) and the on-shell

closure of the algebra of the associated (2,0) supersymmetry transformations, the new condition

$$\nabla_i t^a - I^j{}_i \nabla_j s^a = 0 \tag{7.5}$$

is satisfied. The closure of the algebra of (2,0) supersymmetry transformations (7.3) has been studied before for the special case where I is a complex structure [18, 19]. Finally, the Nijenhuis transformations (7.4) leave the action invariant and the algebra of transformations (7.2) and (7.4) closes provided that, in addition to the conditions (2.11)-(2.13) and (3.10) given in sections two and three, the following conditions are satisfied:

$$N^k_{ij} \nabla_k s^a = 0, \qquad N^k_{ij} \nabla_k t^a = 0 \ . \tag{7.6}$$

The conditions (7.6) are not independent but they are integrability conditions for (7.5) and the condition that $F(A)$ is a (1,1)-form with respect to the almost complex structure I. The algebra of transformations (7.2) and (7.4) of the massive (2,0)-supersymmetric sigma model is isomorphic to the algebra of transformations (3.3) and (3.6) of the massless model. Finally, it is straightforward to extend the above results to massive (2,2)-supersymmetric sigma models with Nijenhuis symmetries.

7.2. Topological models

One way to associate a topological sigma model to a supersymmetric one is by twisting the supersymmetry algebra of the latter [20]. The traditional way to twist the supersymmetry algebra of a sigma model is to define a new Lorentz generator which is the sum of the original Lorentz generator of the theory with a linear combination of the generators of the abelian subgroup of the automorphism group of the its supersymmetry algebra that rotate supersymmetry charges but leave the rest of generators of the algebra invariant. Then new Lorentz weights for the generators of the supersymmetry algebra are assigned with respect to this new Lorentz

generator. Provided that the new Lorentz generator is chosen appropriately, some of the supersymmetry charges of the sigma model transform under the new Lorentz transformations as scalars and the Poisson brackets of certain linear combinations of them either vanish or close to a commuting central charge of the algebra with new Lorentz weight zero. Such charges that are anticommuting, Lorentz scalars and their Poisson brackets either vanish or close to a central charge can be thought as BRST charges and these are the charges that generate the symmetries of the associated topological model.

The method described above to associate a topological sigma model to a supersymmetric one does not seem to be applicable in the case of (2,0)-supersymmetric sigma models with Nijenhuis symmetries because rotations of the supersymmetry charges of (1.2) that leave the rest of the charges invariant are not automorphisms of the algebra. An alternative way to twist the algebra (1.2) is to use the automorphism (5.13) and define a new Lorentz generator $L' = L - \frac{1}{2}R$. It is important though to note that the automorphism R is not realised by transformations on the fields of the (2,0)-supersymmetric sigma model with Nijenhuis symmetries and therefore the relation between the (2,0)-supersymmetric sigma model and its topological version is not as clear. Nevertheless, all the charges of the (2,0) supersymmetry algebra (5.10) are Lorentz scalars with respect to L' and the non-vanishing Poisson brackets of the twisted (2,0) supersymmetry algebra are

$$\{S^{{}^0}, S^{{}^0}\} = 2T, \qquad \{S^1, S^1\} = 2W, \qquad \{S^{{}^0}, S^1\} = 0 \ , \tag{7.7}$$

where $S^{{}^0}, S^1$ are anti-commuting charges and T, W, $(W = T + N)$, are commuting ones. This topological algebra is similar to the topological algebra that one gets in the equivariant version of topological sigma models, ref. [20], and similar methods can be applied to construct models. The above topological twisting of the (2,0) supersymmetry algebra can be easily generalised to twist (2,2) one.

8. Summary

Sigma models with $N = 2$ supersymmetry have been studied extensively in the literature because of their applications to string compactifications, integrable systems and the novel renormalisation properties of some of their couplings. Most of the effort so far, with some exceptions, has been concentrated to investigate those $N = 2$-supersymmetric sigma models that their algebra of charges is isomorphic to the standard one (1.1). These models have target spaces that are complex manifolds. However we have found a new class of (2,0)-supersymmetric two-dimensional sigma models with target spaces almost complex manifolds extending similar results for (2,2)-supersymmetric sigma models. The supersymmetry algebras of these (2,2)- and (2,0)-supersymmetric sigma models close on finite number of generators and apart from the supersymmetry charges, the energy and momentum, they contain other charges of Lorentz weight one that generate new symmetries of the sigma model action associated to the Nijenhuis tensor of the almost complex structures. We have shown that the supersymmetry algebras of sigma models with Nijenhuis symmetries are not isomorphic to standard one (eqn. (1.1)). The supersymmetry algebras of (2,2)- and (2,0)-supersymmetric sigma models with target space an almost complex manifold are *not* a counter-example to the Haag-Lopuszanski-Sohnius theorem because this theorem deals with the structure of four-dimensional supersymmetry algebras. Supersymmetry algebras isomorphic to the one of (2,2)- and (2,0)-supersymmetric sigma models with target spaces almost complex manifolds have been have been discussed before in the context of string theory. In the string case, however, the space-time is taken to be flat but topologically non-trivial and the additional Lorentz weight one central charges are associated to winding numbers [21]. The massive (2,0)- and (2,2)-supersymmetric sigma models with target spaces almost complex manifolds have been also considered.

The anomalies in the symmetries of the (2,0)-supersymmetric sigma models with target space almost complex manifolds have been investigated. It can be arranged for anomalies to occur in the frame rotations of the sigma model man-

ifold, the gauge transformations of connection coupling of the fermionic sector, the (2,0)-supersymmetry and the Nijenhuis transformations. The cancellation of these anomalies has been studied, as well, and it has been found that some models are anomaly free like for example (2,0)-supersymmetric sigma models on group manifolds.

The automorphism group of the supersymmetry algebra of (2,0)- and (2,2)-supersymmetric sigma models with target space almost complex manifolds does not contain an element that rotates the supersymmetry charges but leaves the rest of the charges invariant in contradistinction to the automorphism group of the standard supersymmetry algebra (1.1). However it has been found that there is another automorphism that rotates the supersymmetry charges and the two Lorentz weight one charges. This automorphism cannot be realised by field transformations of the associated sigma model but it can be used to twist the supersymmetry algebra. The twisted supersymmetry algebra is similar to the topological algebra that one gets in the context of equivariant topological sigma models.

All (2,0)- and (2,2)-supersymmetric massless sigma models with off-shell supersymmetry and algebra of charges isomorphic to (1.1) admit a manifest (2,0) and (2,2) conventional superfield formulation in terms of constrained superfields, respectively. A superfield formulation has also been constructed for some (2,2)-supersymmetric sigma models with on-shell supersymmetry and target spaces complex manifolds [5]. We have shown that there are off-shell (2,0)-supersymmetric sigma models with target spaces almost complex manifolds. However a superfield formulation of the these models similar to that of ref. [4] for (2,0)-supersymmetric sigma models with target spaces complex manifolds does not seem to be applicable. We leave to the future the study of this problem.

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